The choice wave: An alternative description of consumer behavior

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ABSTRACT

The question of proper incorporation of consumers that deviate from rationality as defined by classical economics has been considered in earnest since the 1970s. This study proposes the concept of a choice wave, a probabilistic component within a utility function. The choice wave describes consumers who maximize utility every time they make a consumption choice, even though the choices may be different over time. The approach permits the variations in consumer choice such as those mentioned in prior work to be considered economically rational. When applicable, this theory may permit econometric estimations that are more insightful with respect to consumer behavior and psychology.

Keywords: choice, rationality, consumer behavior, quasi-rationality, econometrics

1. INTRODUCTION AND THEORY

Are consumers always rational? Is what we consider a deviation from rationality really irrational? Economic research that was well underway by the 1980s centered on consumer behavior that deviated from the classical notions of economic rationality sought to explain why consumers behaved as they did. Indeed, humans often behave differently from the way they are described by economists (Rabin 1998).

Russell and Thaler (1985) termed a subgroup within a population that does not behave according to economic rationality "quasi-rationals" and demonstrated that failure to include such consumers (if there were enough of them) in a model rendered it inaccurate. Akerlof and Yellen (1985) dealt with the effects of equilibrium of consumers who did not maximize utility, even including small deviations from classical utility-maximizing behavior. Issues of incomplete information may also account for variations in consumer behavior (Nayga 2001). Kahneman and Tversky (1979) devised a weighted expected utility function to account for consumer variations in behavior. Other studies about deviations from classical rationality and variation in utility maximization have continued. For example, Scott and Yelowitz (2007) dealt with status goods, and McFadden (2005) studied the relationship between cognitive psychology and economics, suggesting that behavioral aspects of the consumer should be accounted for in measures of utility. The Random Utility Model sought to consider the possibility that there existed consumers whose utility differed from classical utility, and that difference was treated as a random term. In Johnson (2007), utility functions based on probability waves determined by consumer revealed preference were used to split a data set by consumer revealed preference. This resulted in statistically different price and income elasticities of demand between two groups of consumers. The present study follows this conceptual framework in which the variations in consumer choice such as those mentioned in prior work may actually be considered economically rational. When applicable, this theory may permit econometric estimations that are more insightful with respect to consumer behavior and psychology.

For simplicity, the straw man in this framework is someone who allocates income between two goods and has a continuous spectrum of expenditure combinations conditional on that bundle of two goods. Assume the consumer may choose differently over time. Kahneman and Thaler (1991) explained this variation over time as due to adaptation and experience. However, here there will be no such restrictions placed on the reason why choice may change over time. The consumer may choose one combination at a certain moment in time to maximize utility and a completely different combination at another moment in time, also to maximize utility. There is a probability associated with each possible combination of expenditures on the two goods. This consumer's choice is assumed to be probabilistic, not random.

In this framework, the consumer is able to "oscillate" in some manner between choices in between making decisions. This means that the economist as the observer does not know what choice will be made until the consumer actually chooses. The consumer's mind "oscillates" between all the utility maximizing combinations within the bundle and then chooses. Only when the consumer makes a choice does the outside observer (the economist) know what maximized the individual's utility. As such, this probabilistic component can be considered a "probability wave" (Johnson 2007).

Consider making a choice between beer and chips at the store. You decide today how much beer and how much chips to get by observing the two products and determining what will maximize your utility at that moment. The expenditure combination today may be completely different from expenditure combination you choose tomorrow. This does not inherently imply about the amount of time spent in choosing amounts of beer and chips. There is also no assumption as to the temporal stability of the probability, but there is an assumption that the possibility exists for the consumer's choice to vary over time in a probabilistic manner and still maximize utility. That is, while the consumer might have a preferred amount of expenditure on each of the two goods, there is assumed to exist a probability of some value that other expenditure levels may be chosen and still maximize utility. This preserves rationality at each decision point, despite the potential for temporal variability of choices.

Note, however, that this assumption does not mean that the consumer is maximizing utility over all time. There exists the possibility for the consumer to make a choice that is, in the long run, not utility-maximizing, but does maximize utility at the moment the decision is made. There is no assumption that the consumer's choice is always "wise" in the long-run; simply that it maximizes utility at the moment the consumer decides the make a choice. This is unique to the individual and is based on a complex set of internal cognitive characteristics (Thaler 1994; McFadden 2005).

Fig. 1 provides a visual depiction of choice in this framework, where the consumer allocates income between the two goods in the bundle *A* and *B*. This diagram looks very much like the classical model, except for one important difference. In the left-hand frame, the indifference curve is "floating" along the budget constraint. That is, there is a probability associated with each potential position of the indifference curve. Only when the consumer chooses does the indifference curve take on its normal appearance, being concretely at the position corresponding to the expenditure choice of the consumer. This is shown in the right-hand portion of Fig. 1.

One additional important assumption is the potential for the choices of a consumer and one or more other consumers being completely or partially uncorrelated. This can be exemplified by considering if two consumers go into the store to buy beer and chips, there is certainly no reason to assume that they will necessarily both have the same probabilistic function and make the same expenditure choice, even if they choose at the same time. In fact, what influences the decision-making process of one consumer may or may not have any effect whatsoever on the other consumer. Therefore, in this conceptual framework, two consumers can make different decisions at the same time, both maximizing utility, and have utility functions that are not the same. These distinct utilities are different because they have different probabilities.

The consumer's utility in this framework needs to satisfy the conditions given by the assumptions, i.e., it must be continuous, probabilistic, vary in a non-random manner over time, always lead to temporal utility maximization, and permit the existence of one or more individuals who chose according to unique probability functions. This utility function is termed a "choice wave." Using the choice wave may permit better prediction of market behavior by including *post facto* the presence of various effects of cognitive processes of the consumer.

2. THE CHOICE WAVE

The previously-mentioned Random Utility Model (RUM) is an existing conceptual model that acknowledges that there may be variation in utility differing from the representative consumer within the population. In the RUM, variations from systematic (classical) utility are treated as random with utility given as $U_e = V_e + \varepsilon_e$. The systematic utility is V_e, and the random component is the error term ε_e. A choice wave continues this notion with the modification that utility is purely probabilistic. The classical and random terms are replaced with a probabilistic component having an underlying assumption that the probability of choosing any expenditure level e at any given decision point is given by $P(e \mid t) = P\left(U_e(e)\mid_t = MaxU^*\right)$. This probability expresses the notion that the probability of choosing the expenditure value e is equal to the probability that the utility of that expenditure conditional on time, i.e., at the decision point, maximizes utility. Maximum utility at the decision point is denoted as $MaxU^*$. Now, at a decision point, $P(e) = 1$ if the expenditure level *e* maximizes utility, and $P(e) = 0$ if it does not maximize utility. What about the probability when not at a decision point?

Recall the assumptions of the choice wave given earlier. The probability of the expenditure level *e* being chosen when *not* at the decision point is not a simple 0 or 1 option. There is a probability of all possible expenditure levels in between decision points, as it is unknown what the consumer will choose. The probability in between decision points is given as $P(e,t) = P\left(U_e(e,t) = MaxU\right)$. Maximum utility at any point in between decision points is denoted as $MaxU$. Note that in this probability, a time component is included. This acknowledges that the probability is a probability wave and can vary over time in between decision points. This is the notion that the choice that maximizes utility at $t = t_0$ is not automatically the choice that maximizes utility at $t = t_1$. Combining this "overall" probability with the probability at a decision point, the elemental definition of the choice wave, $\psi(e)_t$ over all time is given by:

$$\psi(e)_t = \begin{cases} P\left(U_e(e \mid t) = MaxU^*\right), & \text{at the decision point;} \\ P\left(U_e(e,t) = MaxU\right), & \text{otherwise.} \end{cases} \tag{1}$$

Fig. 2 gives a graphical depiction of (1). In the left-hand panel, there is an arbitrary probability wave given. In the right-hand panel, the probability wave has collapsed to a spike such that the probability is 1 at the utility-maximizing expenditure at the decision point, 0 elsewhere.

Whatever the functional form of the choice wave, it must satisfy the probability conditions in (1), as well as the other assumptions given in the introduction. The functional form of the choice wave depends on two equations, viz., the Choice Wave Primal Equation and the Market Potential Function. Both of these must lead to wave functions that satisfy the assumptions as well as (1). Sections 3 and 4 introduce the Market Potential Function and the Choice Wave Primal Equation, while section 5 shows how the choice wave evolves from the Market Potential Function and the Choice Wave Primal Equation.

3. MARKET POTENTIAL FUNCTION

The Market Potential Function (MPF) is used to describe the "potential" for purchase that a consumer has, i.e., the allowable expenditures. This is the choice wave framework version of a budget constraint.

Begining with the notion that the consumer has income *I* that *may* be allocated between two goods, *A* and *B*,[1] and has the capacity, or the potential, to spend none of *I*, all of *I*, or

[1] This applies to larger numbers of goods, but the two-good case is used throughout the paper for clarity of explanation.

something in between. A simple version of the MPF for expenditure on the i^{th} good that satisfies this condition is given by:

$$MPF\left(e_i\right)=\begin{cases}0, & 0<e_i<I;\\ \infty, & otherwise.\end{cases} \quad (2)$$

When *MPF=0*, the consumer has the potential to spend. When $MPF=\infty$, the consumer cannot spend. See Fig. 3. Think of the MPF like a wall of infinite height. The consumer cannot get over it, so the consumer cannot spend. In (2), the consumer has capacity to spend within their income range. The MPF simply states the budget constraint in choice wave terms. Further, the consumer allocates all of income *I* between *A* and *B*, and at a decision point, all income will be spent. What varies between the decision points is the allocation of *I* between each of the two goods in the bundle.

4. CHOICE WAVE PRIMAL EQUATION

The Choice Wave Primal Equation (PE) is, in its elemental form, an equation containing the choice wave ψ such that when it is solved for ψ the resulting equation for ψ will satisfy the underlying assumptions and requirements of a choice wave. A sufficient form of the PE that yields ψ satisfying the requirements is given by:

$$-U\left(e\right)\frac{d^2\psi\left(e\right)}{de^2}=H\psi\left(e\right) \quad (3)$$

In (3), $U\left(e\right)$ is the "actual" utility realized from consumption at the decision point, i.e., the classical utility. In addition to this actual utility, there is also potential utility. This is the utility in between decision points that a consumer has the potential to realize, yet does not realize since no decision is made at those times. The variable *H* is total utility, defined as the sum of actual and potential utility on the good in question, as well as utility from consumption of other goods not in the model. The total utility on the right-hand side of (3) is weighted by its probabilistic component, ψ. The actual utility, $U\left(e\right)$, is weighted by the second derivative of ψ with respect to expenditure. Under the assumption of diminishing marginal utility, $\frac{d^2\psi\left(e\right)}{de^2}<0$. The time subscript has been dropped for simplicity from the choice wave. See Appendix 1 for the derivation of (3).[2]

5. DERIVING THE CHOICE WAVE

The choice wave is obtained by solving (3) for ψ. See Appendix 2 for the complete solution for (4) below.

[2] The form of the Choice Wave Primal Equation in (3) is analogous to the Schroedinger Wave Equation in quantum mechanics. The actual utility portion of (3) is analogous to the kinetic component of the Schroedinger Wave Equation, and the right-hand portion is analogous to the Hamiltonian of the Schroedinger Wave Equation.

$$ {}_{T}\psi_{g}\left(e\right)=\begin{cases}\sqrt{\dfrac{2(g+1)}{gI}}\sin\left(\dfrac{g\pi e}{I}\right), & g \text{ odd;}\\ \sqrt{\dfrac{4}{gI}}\sin\left(\dfrac{g\pi e}{I}\right), & g \text{ even.}\end{cases} \tag{4} $$

The postscript *g* is a "wave state." The range of *g* is from 0 to infinity. The implication of this is that now a function exists that can uniquely define each consumer probabilistically in such a way that the choice probabilities of that consumer are completely unique. The choice wave of one consumer exists only in the space of the consumer and not in the space of any consumer. Similarly, the choice waves of all other consumers exist only in their space, and not in the space of any other consumer. Changing wave state yields a choice wave that is orthogonal to every other choice wave. Indeed, solutions are inherently orthogonal. This is key, in that it establishes the consumer in one state exists in their own choice space and not in the choice space of any other consumer. Having separate choice space for each consumer is the most complete description of the market. Yet, this is less useful in practical terms for estimation purposes.

Consumers may indeed exist in each others space at least somewhat. Therefore, an individual's effective choice wave, which comprises the PDF of that consumer, may be actually a linear combination of several choice waves, some of which may be shared by other consumers. If they share one or more choice waves with a representative consumer, and their expectation value of expenditure is not statistically different from that of the representative consumer, then it is reasonable to assume that consumers meeting this requirement may be approximated by the choice wave of the representative consumer. If there exist more than one group of consumers for which the preceding is true within each group, and the members of one group do not exist in the choice space of the members of any other group, then the market is said to have more than one Consumer Type. The form of the choice wave given in (4) is "Consumer Type specific." The pre-script *T* is a marker used to identify the Consumer Type.

In addition, the choice wave in (4) allows for variability over time, even without an explicit time component (see Appendix 1 and the derivation of the Primal Equation). It allows *e* to take on any value as allowed by the MPF. The assumptions and requirements for an effective choice wave have been satisfied.

Note that this is not the only functional form that could be used. It is one form that meets the requirements. The ultimate choice wave would be one that completely and accurately captures all internal cognitive processes of the consumer. Without the ability to see inside the consumer's mind, the selection of choice wave is a matter of choosing a reasonable function that satisfies the underlying assumptions.

As previously stated, the choice waves have the potential to generate a set of utility functions for each consumer in the market, and this clearly is not practical. Though this has been treated in brief, the practical and statistical application of the choice wave to a market will be discussed further in section 6.

6. PRACTICAL AND STATISTICAL APPLICATION

In its most expansive use, a wave function can be assigned to each consumer, provided that it accurately captures their behavior and none of the traits of any other consumer. This is not very practical for empirical use. In practical settings, consumers may actually exist in each others space somewhat, i.e., there is some overlap in tastes and preferences, cognitive processes, etc.

Since each choice wave represents completely different consumers, in real life the consumers may be represented by composite choice waves that are linear combinations of choice waves.

Classical theory states that choice of those in a population may be thought of as the choice of a single representative consumer. In choice wave theory, there may be multiple representative consumers, one for each "consumer type."

A consumer type is a group of consumers that all choose statistically the same as the Type Representative Consumer (TRC). If the consumers in this group choose according to composite choice waves that are linear combinations of basic choice waves, then there must be an average composite choice wave for that group. This composite choice wave may be a linear combination of various basic choice waves, or it may be the case that a basic (single) choice wave satisfactorily represents the average of the group. The necessary conditions for two or more groups of consumers to comprise distinct consumer types are that their expectation values are statistically different and the choice wave of the TRC of one type cannot be the same as or comprise any of the basic choice waves of the TRC in any other consumer type.

The advantage to this approach empirically is the orthogonality of the choice waves of each consumer type's TRC. If two or more groups of consumers can be expressed by distinct choice waves, the orthogonality of the choice waves implies that the two groups are completely distinct. If they are distinct, then they may be modeled separately. For example, Johnson (2007) successfully split a data set comprised of scanner data among various US metropolitan areas into two statistically different consumer types using a variation of the Almost Ideal (AI) demand system based on a probability wave utility function. The distinct consumer types were shown to have different, yet plausible price and income elasticities of demand, both from each other and from the elasticities when the entire data set was modeled together.

What about the case when all consumers have composite wave functions such that they may not be broken into distinctive groups with different expectation values? In this case, the choice wave theory becomes almost the same as classical theory. The only exception is that the probabilistic nature of the choice wave still exists. Now it only applies to a single representative consumer as in classical theory. In expectation value, the choice wave result for a single group will be the same as the classical economics result.

7. CONCLUSIONS

This study proposes a conceptual and mathematical framework in which the presence of cognitive psychological differences, both among consumers and temporally within the mind of an individual consumer may be accounted for in a probabilistic way. The choice wave was introduced as the probabilistic component of utility underlying consumer choice over time. The choice wave theory has several key assumptions that must be present in order for a given choice wave to provide a valid description of consumer behavior. Choice waves must allow all utility-maximizing choices and no choices that do not maximize utility. This allows different choices to be made at different times, while still maximizing utility. In between decision points, there must be a probability associated with each of these choices, with the maximum probability at the consumer's most preferred choice, i.e., the expectation value. At the time the consumer makes a decision, the choice wave must "collapse" so that the probability of making the choice is 1, and the probability of not making the choice is 0. Lastly, all choice waves must be orthogonal. This permits the possibility of multiple consumer types that do not exist in each other's space and hence can be modeled separately. This is the key empirical implication, and there is no reason to assume that it cannot be applied to modify any standard econometric model. Further research will center on development of a choice wave demand function and expansion of empirical possibilities.

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Appendix I. Derivation of the Choice Wave Primal Equation (PE)

In the framework of choice wave theory, the consumer's goal is to maximize utility *at every decision point*, i.e., at every time they make a consumption choice. Thinking of utility maximization in this way, then, it has both a consumptive component (expenditure) and a temporal component. This may be expressed generally as $u(e,t)$. Because the exact moment of their choice in time, i.e., the decision point, is unknown before the choice is made, there exists a probability of utility given an expenditure level. This utility is, then, a probabilistic function.

It is now necessary to derive a form of $u(e,t)$ that will lead to a choice wave ψ that satisfies the assumptions of choice waves. A satisfactory means to accomplish this is to let $u(e,t)$ represent solutions to the second order differential equation as follows:

$$U(e|t) = \alpha \frac{\frac{\partial^2 u(e,t)}{\partial t^2}}{\frac{\partial^2 u(e,t)}{\partial e^2}} \qquad \text{(A1.1.)}$$

In Eqn. A1.1., the component $U(e|t)$ is the "actual utility, i.e., the revealed preference utility, or the utility the consumer receives from their consumption level e at the moment of decision. It is conditional on *t*, the moment of decision.

In Eqn. A1.1., actual utility at the moment of decision is assumed to be proportional by some factor α to the ratio of the derivatives of marginal utility with respect to time and expenditure. Assume for purposes of example that $\alpha < 0$, $\frac{\partial^2 u(e,t)}{\partial t^2} < 0$, and $\frac{\partial^2 u(e,t)}{\partial e^2} < 0$ at each decision point. This imposes concavity on the utility functions. If the time second derivative value $\frac{\partial^2 u(e,t)}{\partial t^2}$ is high relative to the expenditure second derivative $\frac{\partial^2 u(e,t)}{\partial e^2}$, then $U(e|t)$ will be higher, as there are diminishing marginal returns to waiting longer to make a decision. If on the other hand $\frac{\partial^2 u(e,t)}{\partial e^2}$ is high relative to $\frac{\partial^2 u(e,t)}{\partial t^2}$, then $U(e|t)$ will be lower due to diminishing marginal utility.

Now separate Eqn. A1.1. into an expenditure component and a time component. By separation of variables, let $u(e,t) = \psi(e) f(t)$, where $\psi(e)$ is the choice wave and $f(t)$ is a function over time. Applying this separation of variables to Eqn. A1.1. yields Eqn. A1.2.

$$f(t)\frac{d^2\psi(e)}{de^2} = \frac{\alpha}{U(e)}\psi(e)\frac{d^2 f(t)}{dt^2} \qquad \text{(A1.2.)}$$

Now a form of the time equation must be assumed. Since it represents the contribution of *when* a consumer makes a choice, and hence is practically impossible to know in its entire form without knowing what is going on inside the mind of the consumer at every moment, a simple cyclical function will be used. A cyclical function is chosen because it allows for variation over time around a mean. This will prove useful for empirical estimation.

Let $f(t) = \exp(i\omega t)$, which is an elemental cyclical function with period = ω. Economically, the period is the temporal stability of the utility function. From this follows Eqn. A1.3.

$$\frac{d^2\psi(e)}{de^2} = \frac{-\alpha}{U(e)}\omega^2\psi(e) \qquad \text{(A1.3.)}$$

Now define H^* as the "total utility" of the system such that $H^* \equiv U(e) + U_p(e)$. The second term, $U_p(e)$, is potential utility. Potential utility is defined as utility that could be realized from consumption of a good at a given moment in time, but is not realized. So, the total utility is the utility the consumer realizes at the moment of decision plus the utility they could have realized, but do not. Before a decision is made, actual utility is zero and all utility is potential utility. If a consumer has utility they could have realized at the decision point, but do not, then they are economically irrational because they are not maximizing utility. However, in the choice wave framework, the consumer is assumed always to maximize utility at the moment the consumption decision is made.

Now let $\omega^2 = \beta U(e)$. The factor β is a function or coefficient that may be thought of as a factor pertaining to consumer preferences. It gives the relationship between the period, i.e., the variability of consumer preferences over time in between decision points (which cannot be observed) and actual realized utility at the decision points (which can be observed). A key important point here is that the utility is not random, is probabilistic. There is some relationship between the consumer's mind varying from choice set to choice set in between decision points and the revealed choices at the decision points. This definition and the definition above of H^* may be combined to give Eqn. A1.4.

$$\omega^2 = \beta\left[H^* - U_p(e)\right] \qquad \text{(A1.4.)}$$

Combining Eqns. A1.3. and A1.4. yields the following:

$$\frac{d^2\psi(e)}{de^2} = \frac{-\alpha}{U(e)}\beta\left[H^* - U_p(e)\right]\psi(e) \qquad \text{(A1.5.)}$$

As previously mentioned, there exists the underlying assumption in choice wave theory that, when a consumer makes an expenditure choice at the decision point, the consumer has maximized utility at that point, and so is rational. If the consumer has maximized utility, then all potential utility at that point has been "converted" into actual utility, so potential utility must be zero. Letting $U_p(e) = 0$ in Eqn. A1.5. yields:

$$\frac{d^2\psi(e)}{de^2} = \frac{-\alpha\beta}{U(e)}H^*\psi(e) \qquad \text{(A1.6.)}$$

As β is an unknown factor, define $H = \alpha\beta H^*$. This yields Eqn. A1.7. from Eqn. A1.6.

$$-U(e)\frac{d^2\psi(e)}{de^2} = H\psi(e) \qquad \text{(A1.7.)}$$

This is the Choice Wave Primal Equation (PE). Eqn. A1.7. is solved for the choice wave $\psi(e)$ in Appendix 2. The choice waves derived from A1.7. have only the expenditure component explicitly, and no explicit probabilistic terms. Yet, the various assumptions of consumer behavior remain underlying any choice wave derived from A1.7. QED.

Appendix II. Derivation of the Choice Wave

II.1. Initial Derivation

Solving the Choice Wave Primal Equation, Eqn. A1.7, for ψ yields the choice wave. This is a standard solution to a second-order differential equation.[3] Following Johnson (2007) rewrite the PE, Eqn. A1.7. as follows:

$$\frac{d^2\psi(e)}{de^2} = -k^2\psi(e) \qquad \text{(A2.1.)}$$

The general solution to the second-order differential equation A2.1. is given by:

$$\psi(e) = A\sin(ke) + B\cos(ke) \qquad \text{(A2.2.)}$$

To obtain values for *A* and *B*, the boundary conditions are considered. Because of the assumption of continuity of the choice wave over all space, $\psi(0)=0$. Therefore, $B=0$. Also by the assumption of continuity, $\psi(I)=0$. Therefore, *kI* either must equal the trivial solution of 0, or $kI = n\pi$, where *n* is an integer. Applying these boundary conditions to Eqn. A2.2. yields the following:

$${}^{\dagger}\psi(e) = A\sin\left(\frac{n\pi e}{I}\right) \qquad \text{(A3.3.)}$$

The dagger pre-script in Eqn. A3.3. has been added as a notation to indicate that this choice wave has not yet been normalized and the coefficient *A* has not yet been calculated. To calculate *A*, two properties of choice waves will now be introduced.

II.2. Choice Wave Property No. 1 – Probability Relationship

The probability density function $\phi(e)$ in a system modeled using choice waves is the squared modulus of the choice wave, i.e., $\phi(e) = |\psi(e)|^2$. The squared modulus is the complex conjugate of the choice wave multiplied by the choice wave itself, i.e., $|\psi(e)|^2 = \psi(e)^*\psi(e)$. As this particular function has no complex component, the PDF is given by $\phi(e) = [\psi(e)]^2$.

II.3. Choice Wave Property No. 2 – Normalization Property

A choice wave must only allow expenditure choices that are possible, given the budget constraint. A normalized choice wave must meet the following criterion:

$$\int_0^I |\psi(e)|^2\, de = 1 \qquad \text{(A2.4.)}$$

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[3] This approach is well-established in quantum mechanics. There, the Schroedinger Wave Equation, a second-order differential equation in physics analogous to the Choice Wave Primal Equation in economics, is solved for a wave function, the quantum mechanical counterpart to the choice wave in economics. See Griffiths 1995, Feynman et al. 1965, and Sakurai 1994.

Eqn. A2.4. states that there is a 100% probability that expenditure choice will lie between 0 and *I*, and by extension, a 0% probability that it will exceed the budget constraint.

II.4. Normalization of the Choice Wave

Choice Wave Properties 1 and 2 are used to calculate the coefficient *A* in Eqn. A2.3. such that the choice wave is normalized and hence preserves the budget constraint. Applying the properties to A3.3. yields:

$$\int_0^I \left|{}^{\dagger}\psi(e)\right|^2 de = \int_0^I |A|^2 \sin^2\left(\frac{n\pi e}{I}\right) de = 1 \qquad \text{(A2.5.)}$$

The solution to this integral is $|A|^2\left(\frac{e}{2} - \cos(2e)\right)\Big|_0^I = |A|^2 \frac{I}{2} = 1$. Therefore, $A = \sqrt{\frac{2}{I}}$. This gives the general normalized choice wave as:

$$\psi_n(e) = \sqrt{\frac{2}{I}} \sin\left(\frac{n\pi e}{I}\right) \qquad \text{(A2.6.)}$$

In Eqn. A2.6., the coefficient and postscript *n* refer to the "wave state." In choice wave theory, the waves exist over an infinite-dimension space, i.e., $0 < n < \infty$ such that $n_i \perp n_j$ for any two wave states *i* and *j*. At this point, each choice wave in A2.6. will yield the same probability density function and the same expectation value. It is necessary to modify the general normalized choice wave to allow for variations among individuals or groups. To do this, Choice Wave Property No. 3 is introduced.

II.5. Choice Wave Property No. 3 – Expectation Value of Expenditure

The expectation value of a choice wave is used to give the expectation value of consumer expenditure. Expectation is expressed by

$$\langle e \rangle_n = \int_0^I e \left|\psi_n(e)\right|^2 de \qquad \text{(A2.7.)}$$

The classical economics expectation value of expenditure is $[e] = \rho_1 e_1 + \rho_2 e_2 + \ldots + \rho_k e_k$, where ρ_i refers to the probability of the i^{th} expenditure over the choice set of *k* possible expenditure choices. The classical expectation value $[e]$ and the choice wave expectation value $\langle e \rangle_n$ are very similar mathematically and become increasingly similar as *k* becomes increasingly large. The classical expectation value is a function of discrete choices of expenditure, regardless of the level of *k*, while the choice wave expectation value is always continuous. The choice wave expectation value is given by:

$$\langle e \rangle_n = \int_0^I \frac{2}{I} \sin^2\left(\frac{n\pi e}{I}\right) de \qquad \text{(A2.8.)}$$

The expectation value as expressed in Eqn. A2.8. is in angular terms due to its wave nature. To put it in useable, practical terms, i.e., fraction of the total expenditure *I*, the following version is used:[4]

$$\langle \bar{e} \rangle_n = \int_0^I \frac{2}{n\pi} \sin^2\left(\frac{n\pi e}{I}\right) de \qquad \text{(A2.9.)}$$

6. Consumer-Specific Choice Waves

Based on Choice Wave Property No. 3, the general normalized choice wave may be modified to represent various individual consumers, as well as groups of consumers by defining choice of the representative consumer of each group. A Consumer Coefficient Function will be multiplied by Eqn. A2.6. to yield the following:

$$_T\psi_g(e) = C_g \sqrt{\frac{2}{I}} \sin\left(\frac{g\pi e}{I}\right) \qquad \text{(A2.10.)}$$

In Eqn. A2.10., the coefficient and subscript *n* has been replaced with *g*. This is a matter of convention so that *n* refers to wave states in the general choice wave, while *g* refers to wave states in the consumer-specific, weighted choice wave. The pre-script *T* is a marker used to identify the consumer type/group that corresponds to the wave state *g*. Each wave state *g* identifies the representative consumer of a Consumer Type.

The form of C_g must be such that the assumptions of choice wave theory are maintained, normalization is not violated, and each wave state yields a different expectation value. Following Johnson (2007), a simple, yet sufficient form of C_g is given as follows:

$$C_g = \begin{cases} \sqrt{\dfrac{(g+1)}{g}}, & g \text{ odd;} \\ \sqrt{\dfrac{2}{g}}, & g \text{ even.} \end{cases} \qquad \text{(A2.11.)}$$

Substituting Eqn. A2.11. into Eqn. A2.10. yields:

$$_T\psi_g(e) = \begin{cases} \sqrt{\dfrac{2(g+1)}{gI}} \sin\left(\dfrac{g\pi e}{I}\right), & g \text{ odd;} \\ \sqrt{\dfrac{4}{gI}} \sin\left(\dfrac{g\pi e}{I}\right), & g \text{ even.} \end{cases} \qquad \text{(A2.12.)}$$

Eqn. A2.12. yields expectation values of expenditure that are different for each wave state *g*. These different wave states may be used to indentify consumers or representative consumers of groups uniquely, or they may be combined linearly, subject to the assumptions of choice wave theory, if necessary to identify consumer behavior accurately. When the choice wave collapses at the moment of decision, the expectation value is the most likely outcome.

7. Orthogonality

[4] See Johnson (2007) for the proof of A2.9.

The final remaining matter is that of orthogonality. Each wave state *g* identifies a choice wave that is orthogonal to choice waves of all other wave states. This is key to the theory and may be shown by the following simple proof, where *K* is a value representing whatever normalization constant and consumer coefficient functions are used.

$$\int \psi_i^* \psi_j de = K\int_0^I \sin\left(\frac{i\pi e}{I}\right)\sin\left(\frac{j\pi e}{I}\right)dx$$

$$= K\int_0^I \left\{\cos\left(\frac{i-j}{I}\pi e\right)-\cos\left(\frac{i+j}{I}\pi e\right)\right\}de$$

$$= K\left\{\frac{I}{(i-j)\pi}\sin\left(\frac{i-j}{I}\pi e\right)-\frac{I}{(i+j)\pi}\sin\left(\frac{i+j}{I}\pi e\right)\right\}\Bigg|_0^I$$

$$= 0$$

In the above proof, the joint probability density of two choice waves of wave states *I* and *j* are shown to yield a probability of zero over a domain allowed by the budget constraint. Therefore they do not exist in each other's space.

Fig. 1

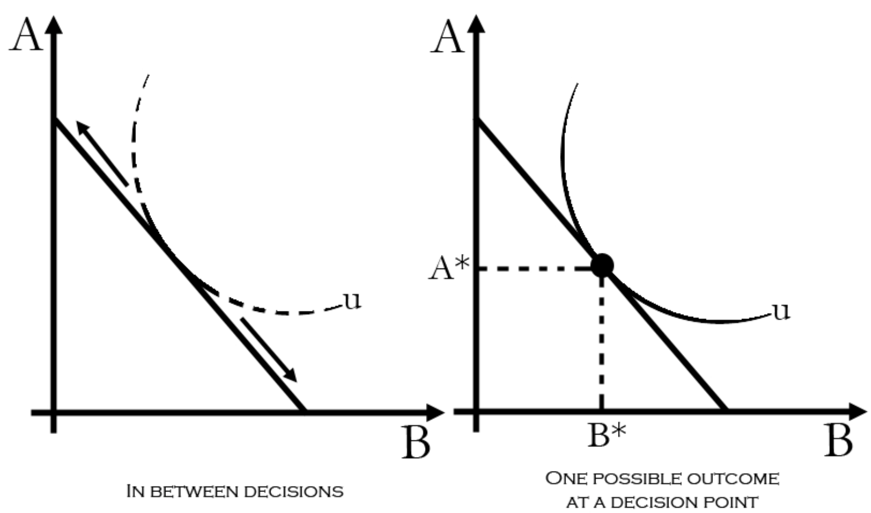
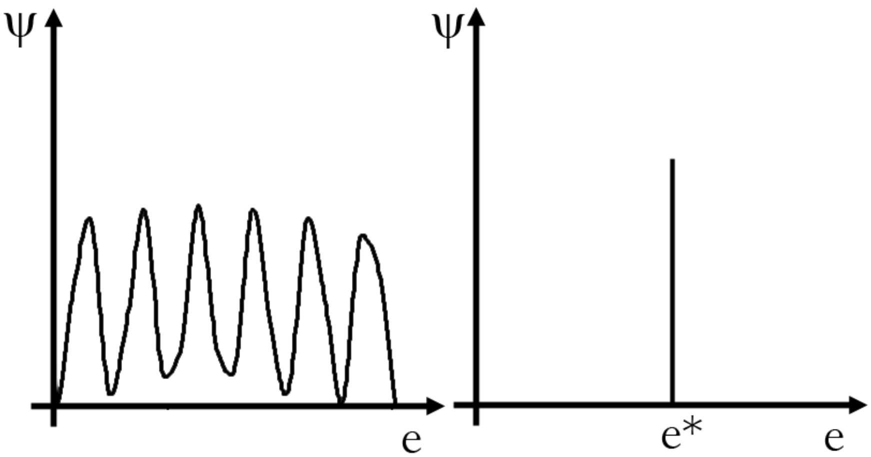


Fig. 2



An arbitrary Choice Wave

A collapsed Choice Wave
At a Decision Point

Fig. 3